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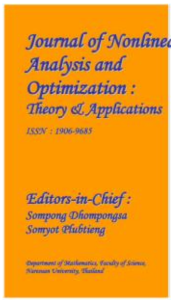

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AN EXAMINATION OF THE STABILITY OF A NONLINEAR SYSTEM USING VARIOUS GROWTH FUNCTIONS

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ABSTRACT: In this study, we will look at a mathematical model that compares the growth of two different species. The current study seeks to investigate the functions of logistic growth and food-constrained growth, two unique growth models. We demonstrated the key features that make semi-trivial and coexistence solutions asymptotically stable. Competitive exclusion of a food-limited population will occur if the population's carrying capacity increases as a result of logistical growth, and the opposite is also true.

Keywords: Equilibria; competition; phase portrait; coexistence.

1. INTRODUCTION

At each and every level of the natural world, there is a robust competition between different species. In order to provide information on the growth rates of two species at low (Malthusian growth) and high (logistic growth) densities, mathematical models are constructed and analyzed. This is done in order to provide information on the growth rates of the two species when they are only impacted by their own populations, which is a phenomena known as intraspecific competition [1]. Interspecific competition is a phenomenon that is being investigated via extended models in order to gain a better understanding of how different species interact with one another when they are competing for limited resources [6]. Competition between species has the effect of lowering the per capita growth rates of both species. According to the research that has been done, the development rules that are the most well-known and widely used were established in [2, 3, 7, 9]. There are a number of biological difficulties that arise when considering competition models between two species that have different development roles [2, 3, 7, 9]. The purpose of this research is to present a competitive model that is founded on the idea of critical population density and adheres to a variety of growth principles. In light of this, the objective of the present research is to develop a fundamental model of competition that is based on the idea that two distinct species compete with one another either interspecifically or intraspecifically at two distinct rates of growth.

The paper is structured in the following manner. We present the auxiliary results that were applied in the process of analyzing the model in Section 2. Additionally, we provide a brief overview of the reaction terms for species u and w, which will be utilized throughout the entirety of the work. Through the process of linearizing the problem that is being considered, Section 3 investigates the equilibrium analysis of semi-trivial steady states and coexistence solutions.

2. METHOD OF ANALYSIS

We examine the following set of nonlinear differential equations while accounting for two distinct species.

$$\begin{cases} \dfrac{du}{dt} = r_1 u(t)\left(1 - \dfrac{u(t)+w(t)}{K_1}\right) \\ \dfrac{dw}{dt} = r_2 w(t)\left(\dfrac{1-\dfrac{u(t)+w(t)}{K_2}}{1+\beta\dfrac{u(t)+w(t)}{K_2}}\right) \end{cases} \quad (1)$$

where the carrying capacities of the species u and w are denoted by K1 and K2, respectively, and the densities of the populations at time t are denoted by u(t) and w(t), respectively. A constant that is positive is denoted by the symbol □, whereas the intrinsic growth rates of u and w are represented by the symbols r1 and r2, respectively. The evaluation of the stability of nonlinear systems can be carried out in a variety of different ways. Two well-known methods for detecting whether or not a nonlinear system is stable are the Lyapunov stability criterion [8, 10] and the LaSalle invariance principle [4, 5]. You can get more information about both methods here. In order to do a mathematical analysis of model (1), we take into consideration the Hartman-Grobman theorem, which is described in more detail below:

Theorem 1. (Hartman-Grobman)

If there are no zero or pure imaginary eigenvalues in the linearization matrix, then the phase portrait of the system that is close to the equilibria (u, v) can be derived from the phase portrait of the linear system by employing a continuous shift in coordinates. This is also possible if the linearization matrix does not contain any zero or pure imaginary eigenvalues.

Featured point of interest 1. This shows that in the event that the matrix does not have any zero or pure imaginary eigenvalues, the stability characteristics of the equilibria (u, v) of the system are similar to those of the equilibrium 0 of the linear system. To be more explicit, this indicates that the matrix does not possess any eigenvalues that are pure imaginary.

Because it is extremely rare for an analytical formula to be able to be formed for the solution of (1), we are often needed to either construct a numerical solution or explore the qualitative behavior of the response. This is because (1) is a problem that is difficult to solve. A qualitative analysis is found to be of tremendous aid when it comes to the building of numerical solutions, as the findings of this study have demonstrated. The study of the solutions in the phase plane (u, w) is something that we find to be of great convenience in any situation that may arise. In the sake of keeping things as straightforward as possible, the two functions that are described below, Stability Analysis of a Nonlinear System with Different Growth Functions, shall be incorporated. After that, the system (1) will be rewritten in the following manner:

$$f(u,w) = r_1 u(t)\left(1 - \frac{u(t)+w(t)}{K_1}\right)$$

$$h(u,w) = r_2 w(t)\left(\frac{1-\dfrac{u(t)+w(t)}{K_2}}{1+\beta\dfrac{u(t)+w(t)}{K_2}}\right).$$

$$\frac{du}{dt} = f(u,w)$$

$$\frac{dw}{dt} = h(u,w) \quad (2)$$

Any solution

$$v(t) = \begin{pmatrix} u(t) \\ w(t) \end{pmatrix}$$

will be understood as a parametrized curve, which is what we mean when we talk about integral curves of the system at this point. The curve in question can be considered an essential component of the system. Through the utilization of the fact that the vector (du/dt, dw/dt) is tangent to the solution curve that is defined by (u(t), w(t), we are able to achieve the outcome that we have been looking for. A graph that conveys the representation of a family of solutions is called a phase portrait. Depending on the requirements of the situation, it can be created numerically or from the direction field. Both methods are viable options. An issue that is related with both the numerical technique and the direction field approach is the requirement that the parameters in (2) be provided. This is a disadvantage that is associated with both of these approaches.

The reason for this is because equilibria of a system occur at points where the coordinates concurrently have a derivative of zero. This gives rise to the aforementioned phenomenon. u(t) = u* and w(t) = w* are the constant solutions that satisfy the nonlinear system of equations f(u, w) = 0 and simultaneously h(u, w) = 0. These points are referred to as the equilibrium, steady state, or critical points. There is another name for these solutions, and that is the optimal solutions. (u*, w*) = (0, 0), (K1, 0), and (0, K2) are the many equilibria that the system (2) is capable of achieving within its parameters. This will be followed by the incorporation of components for the purpose of testing the stability analysis at the equilibrium point.

3. STABILITY ANALYSIS AT THE EQUILIBRIA

Our initial step in this part was to linearize the system so that we could proceed with the analysis. Perform the following reorganization of the functions f(u,w) and h(u, w):

$$f(u,w) = \frac{r_1}{K_1}(uK_1 - u^2 - uw)$$

$$h(u,w) = r_2\left(\frac{K_2 w - uw - w^2}{K_2 + \beta u + \beta w}\right) \quad (3)$$

We begin by doing a calculation in order to locate the linearization matrices at the equilibria.

$$f_u = \frac{r_1}{K_1}(K_1 - 2u - w)$$

$$f_w = \frac{r_1}{K_1}u$$

$$h_u = -\frac{r_2 K_2 w(1+\beta)}{(K_2 + \beta u + \beta w)^2}$$

$$h_w = \frac{r_2(K_2^2 - K_2 u - 2K_2 w + \beta u K_2 - 2\beta uw - \beta u^2 - \beta w^2)}{(K_2 + \beta u + \beta w)^2}$$

The linearization matrices can be obtained by inserting the coordinates of the equilibria into these formulas. This allows us to attain the desired results.

Equilibrium (0, 0): At the equlibrium point (0, 0), four partial derivatives are

$f_u(0,0) = r_1; f_w(0,0) = 0;\ h_u(0,0) = 0$ and $h_w(0,0) = r_2$

Then the linear system

$$\frac{du}{dt} = r_1 u$$

$$\frac{dw}{dt} = r_2 w \quad (4)$$

and the linearization matrix at (0, 0) is

$$M_{(0,0)} = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$$

corresponding eigenvalues of $M_{(0,0)}$ are $\lambda_1 = r_1 > 0$, $\lambda_2 = r_2 > 0$. Put $r_1 = 10$ and $r_2 = 5$ which gives

$\lambda_1 = 10, \lambda_2 = 5$; and the eigenvectors Respectively,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

As a result of the fact that the eigenvalues of the linearization matrix (π,π) are both real and positive, one can arrive at the conclusion that the equilibrium point (0,0) is a repeller. From a biological point of view, when both of the species are present in the same ecological niche, they will be able to repel each other and depart the sub-domain of the habitat until other considerations are taken into account. This will occur until other factors are taken into account. Whenever the system approaches this critical threshold, it is always unstable, as can be shown in Figure 1.

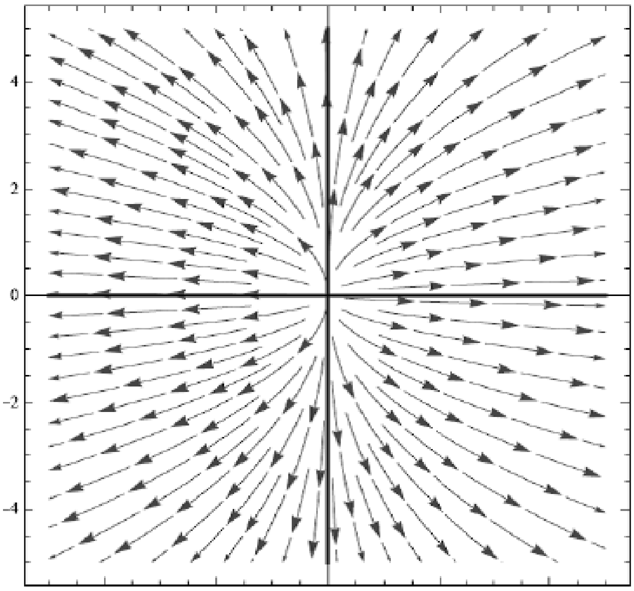


Figure 1. Both the Eigensystem and the phase portrait of the linear system (4) are located close to (0, 0).

Following the calculation of the following terms, we are able to locate the linear system that is in equilibrium (0, K2).

$$f_u(0,K_2)=\frac{r_1(K_1-K_2)}{K_1};\ f_w(0,K_2)=0;\ h_u(0,K_2)=-\frac{r_2}{(1+\beta)}$$

and $h_w(0,K_2) = -r_2/(1+\beta)$

Then the linear system will be

$$\frac{du}{dt}=\frac{r_1(K_1-K_2)}{K_1}u$$

$$\frac{dw}{dt}=-\frac{r_2}{1+\beta}(u+w-K_2) \qquad (5)$$

and the linearization matrix at (0, K2) is

$$M_{(0,K_2)}=\begin{pmatrix} r_1(K_1-K_2)/K_1 & 0 \\ -r_2/(1+\beta) & -r_2/(1+\beta) \end{pmatrix}$$

Eigenvalues of the matrix $M_{(0,K_2)}$ are $\lambda_1 = -r_2/(1+\beta) < 0$ and $\lambda_2 = r_1(K_1-K_2)/K_1$, where λ_2 is positive when $K_1 > K_2$ and negative for $K_1 < K_2$.

As an example,

put $\beta = 2$, $r_1 = 3$, $r_2 = 5$ and for $K_1 = 4$, $K_2 = 3$, we have $\lambda_1 = -5/3$, $\lambda_2 = 3/4$

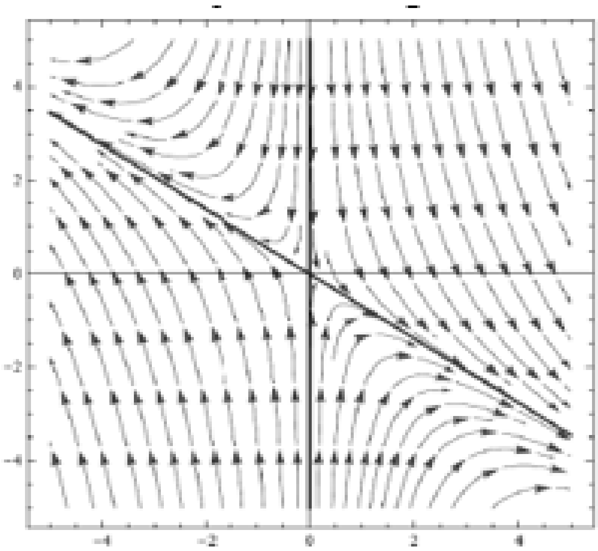
and the corresponding eigenvectors

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} -29/20 \\ 1 \end{pmatrix},$$

based on that information. There is a real value for both of the eigenvalues of the matrix M(0, K2), and the signs of these eigenvalues are the opposite of themselves. Figure 2's left side illustrates that the equilibrium point (0, K2) is an unstable saddle point. This shows that the equilibrium point is unstable. The species u will be able to take advantage of a bigger quantity of natural resources in its oral habitat, which will allow it to outcompete the species w that is following the food-limited growth pattern. This will allow the species u to gain an advantage in the competition. The same is true for K1 = 3, and the same is true for K2 = 4, and the eigenvectors for each of these values are the same.

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} -2/5 \\ 1 \end{pmatrix}$$

The critical point (0,K2) is an asymptotically stable node, as illustrated in figure 2 (right), because the eigenvalues and the critical point are both equal to zero. Because of this, the first population will be wiped out if the second species is able to gain access to the resources more easily than the first population.



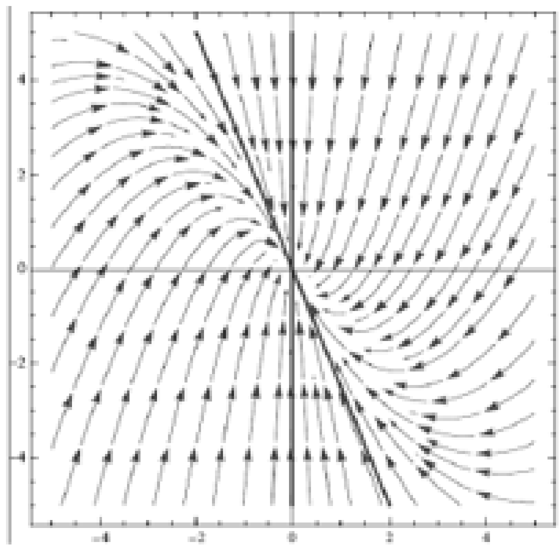


Figure 2. A phase picture of the linear system and the Eigensystem of the system (5)

Equilibrium (K1, 0): The linear system can be obtained by simply applying the following formulae at the coordinates point of the equilibria.

$$f_u(K_1,0)=-r_1;\ f_w(K_1,0)=-r_1;\ h_u(K_1,0)=0 \text{ and } h_w(K_1,0)=\frac{r_2(K_2-K_1)}{(K_2+\beta K_1)}$$

Now given nonlinear system converted to a linear system as

$$\frac{du}{dt}=-r_1(u+w-K_1)$$

$$\frac{dw}{dt}=\frac{r_2(K_2-K_1)}{(K_2+\beta K_1)}w \qquad (6)$$

and the linearization matrix at (K1, 0) is

$$M_{(K_1,0)}=\begin{pmatrix}-r_1 & -r_1\\ 0 & r_2(K_2-K_1)/(K_2+\beta K_1)\end{pmatrix}$$

corresponding eigenvalues of $M_{(0,K_2)}$ are $\lambda_1=-r_1$ and $\lambda_2=R_2(K_2-K_1)/(K_2+\beta K_1)$, here $\lambda_2>0$ if $K_2>K_1$ and $\lambda_2<0$ for $K_2<K_1$.

$$\begin{pmatrix}0\\1\end{pmatrix} \text{ and } \begin{pmatrix}-29/44\\1\end{pmatrix},$$

Based on that information. Both of the eigenvalues of the matrix M(0, K2) are real, and each of them has the opposite sign. Their respective signs are opposite. We have arrived at the conclusion that the equilibrium point (K1, 0) is an unstable saddle point (refer to figure 3, left). This conclusion was reached as a result of the previous statement. Within the framework of the biological concept, it gives the impression that the expansion of the species X is no longer able to be maintained in a particular environment while the competition is going place.

Similarly, for $K_1=3$, $K_2=4$ with same β, r_1 and r_2 as above, $\lambda_1=-5/3$, $\lambda_2=-1$ with respective eigenvectors

$$\begin{pmatrix}0\\-\end{pmatrix} \text{ and } \begin{pmatrix}-2/5\\-\end{pmatrix}$$

Since the eigenvalues λ_1 and λ_2 both are negative, then the critical point $(0, K_2)$ is an asymptotically stable node shown in figure 2 (right). Therefore, if the second species has more accessibility to the resources then the first population goes to extinction.

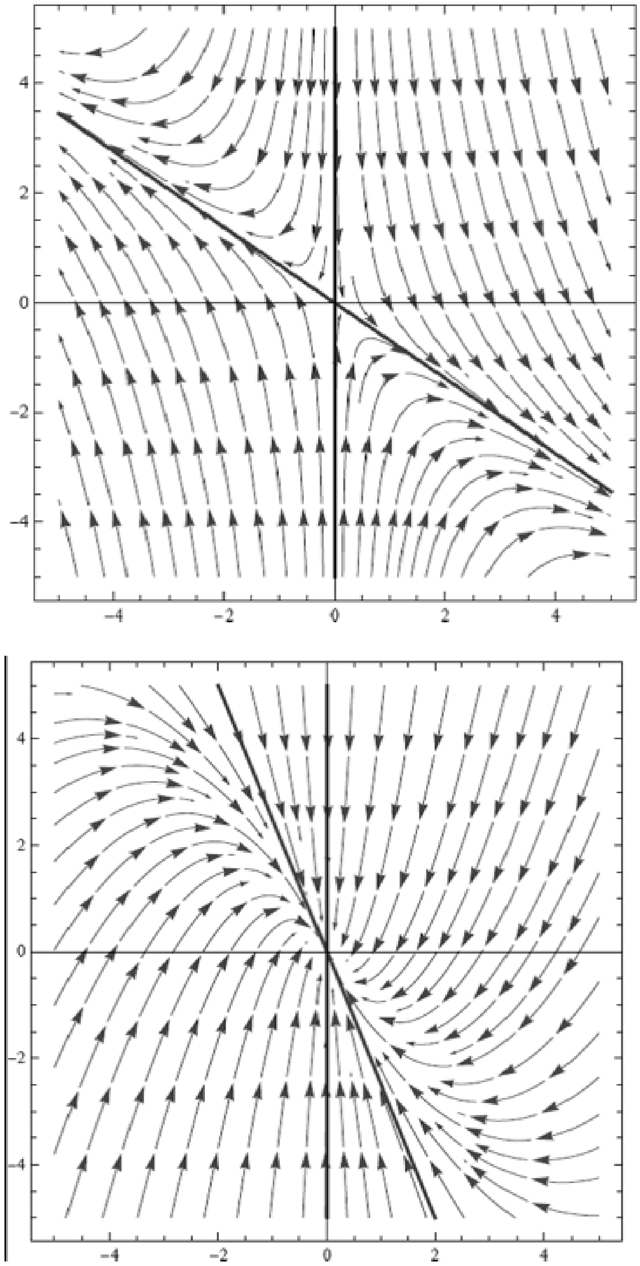


Figure 2. A representation of the eigensystem and phase of the linear system (5), with α equal to 2.

Equilibrium (K_1, 0): The linear system can be obtained by simply applying the following formulae at the coordinates point of the equilibria.

$$f_u(K_1,0)=-r_1;\ f_w(K_1,0)=-r_1;\ h_u(K_1,0)=0 \text{ and } h_w(K_1,0)=\frac{r_2(K_2-K_1)}{(K_2+\beta K_1)}$$

Now given nonlinear system converted to a linear system as

$$\frac{du}{dt}=-r_1(u+w-K_1)$$

$$\frac{dw}{dt}=\frac{r_2(K_2-K_1)}{(K_2+\beta K_1)}w \qquad (6)$$

and the linearization matrix at $(K_1, 0)$ is

$$M_{(K_1,0)} = \begin{pmatrix} -r_1 & -r_1 \\ 0 & r_2(K_2-K_1)/(K_2+\beta K_1) \end{pmatrix}$$

corresponding eigenvalues of $M_{(0,K_2)}$ are $\lambda_1 = -r_1$ and $\lambda_2 = R_2(K_2 - K_1)/(K_2 + \beta K_1)$, here $\lambda_2 > 0$ if $K_2 > K_1$ and $\lambda_2 < 0$ for $K_2 < K_1$. In particular, choosing the parameters $r_1 = 3$, $r_2 = 5$ and $\beta = 3$, then for $K_1 = 3$, $K_2 = 4$, we have $\lambda_1 = -3$, $\lambda_2 = 5/3$ with eigenvectors

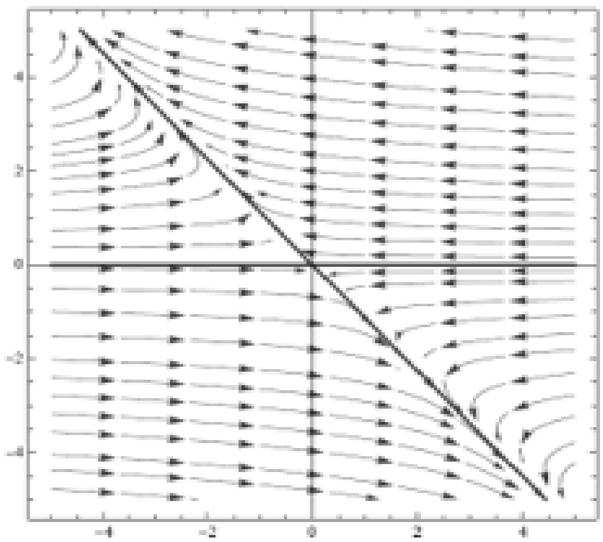
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} -29/44 \\ 1 \end{pmatrix},$$

based on that information. Both of the eigenvalues of the matrix M(0, K2) are real, and each of them has the opposite sign. Their respective signs are opposite. We have arrived at the conclusion that the equilibrium point (K1, 0) is an unstable saddle point (refer to figure 3, left). This conclusion was reached as a result of the previous statement. Within the framework of the biological concept, it gives the impression that the expansion of the species X is no longer able to be maintained in a particular environment while the competition is going place.

In a similar manner, if we set K1 to 4 and K2 to 3, while maintaining the same values for the other parameters mentioned earlier, we will obtain □1 equal to –3 and □2 equal to 5/3, both of which are eigenvectors.

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} -29/34 \\ 1 \end{pmatrix},$$

Since both eigenvalues are negative, we can summarize the critical point $(K_1, 0)$ is an asymptotically stable point shown in figure 3 (right). So, in this circumstance as the species w will be extinct from the given ecological niche.



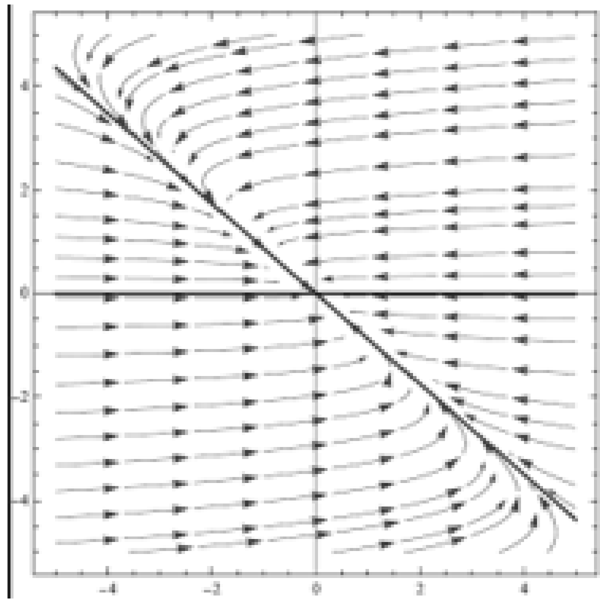
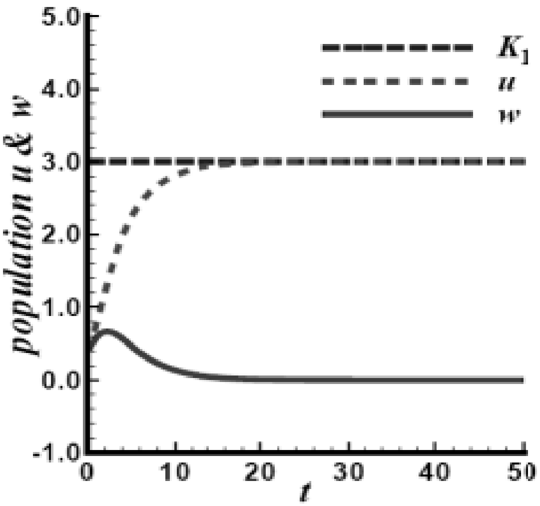


Figure 3. The linear system's Eigensystem and phase portrait are both shown here.

4. NUMERICAL SOLUTIONS

The Runge-Kutta procedures of order four should be taken into consideration for numerical testing in order to solve the system of initial value problem. This is recommended for the purpose of solving the problem.



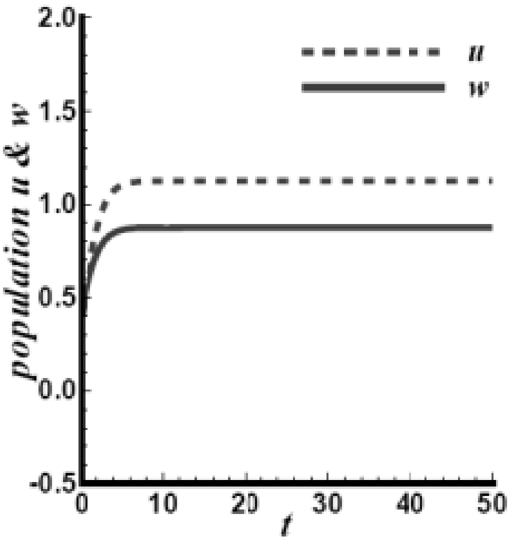


Figure 4 illustrates the numerical solutions to the equations (1) that describe the Logistic-Food constrained growth system. The equations are written with the values α = 0.5, r1 = r2 = 1.0, and (left) K1 = 3.0, K2 = 2.0, and (right) K1 = 2.0 = K2 with initial values u0 = w0 = 2.0. These values are chosen to represent the values of the variables. The carrying capacity K1 is the quantity that corresponds to the solution of the logistic equation when r1 equals r2 equals 1.0 and there are multiple carrying capacities, K1 greater than K2. In contrast, it is worth noting that the solution of the Food-limited equation goes to zero when the value of α is equal to 0.5, as illustrated on the left side of Figure 4. According to the illustration on the right side of Figure 4, one population coexists with the other when K1 equals K2. When the distributions of both populations' resources are equal, despite the fact that their growth functions are different from one another, it is said that the two populations are cooperating with one another.

5. CONCLUSION

Based on the model that was taken into consideration, it can be concluded that carrying capacity is a significant component that plays a role in deciding the outcome of competition between two species in a particular ecological niche. If we make the premise that carrying is the same for both species, then it is possible for them to come together and live together. The parameter α and the intrinsic growth rates are key factors that play a big influence in determining the survival of species. It is worth highlighting that these factors are substantial elements. To add insult to injury, this method can be utilized for certain species in whatever habitat to which they belong.

REFERENCES

1. Gause, G. F. Struggle for Existence, Hafner, New York, 1934.
2. Gilpin, M. E. and Ayala, F. J. Global models of growth and competition, *Proc. Natl. Acad. Sci. USA,* 1973, 3590-3593.
3. Gompertz, B. On the nature of the function expressive of human mortality and on a new mode of determining the value of life contingencies, *Philos. Trans. Roy. Soc. London*, 1825, 513-583.
4. Hale, J. K. Ordinary Differential Equations, John Wiley and Sons, New York, 1969.
5. LaSalle, J. P. The stability of dynamical system, Regional Conference Series in Applied Mathematics. *SIAM*. Philadelphia, 1976.
6. Perko, L. Differential Equations and Dynamical Systems, *Springer, Third edition,* 2008.
7. Smith, F. E. Population dynamics in Daphnia Magna and a new model for population, *Ecology* **4**(1963), 651-663.
8. Strogatz, S. H. Nonlinear Dynamics and Chaos, With Applications to Physics, Biology, Chemistry, and Engineering. Westview Press, Cambridge, 2000.
9. Verhulst, P.F. Notice sur la loi que la populations suit dans son accroissement, *Correspondence Mathematique et Physique*, 1838, 113-121.
10. Wiggins, S. Introduction to Applied Nonlinear Dynamical Systems and Chaos, Springer-Verlag, NewYork, 1983.